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Complete Eulerian-mean tracer equation for coarse resolution OGCMs

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McDougall and McIntosh showed that the adiabatic mesoscale mixing is represented incompletely in the tracer Eulerian-averaged equation (EAE) of coarse resolution OGCMs. We show that completing EAE requires an adequate decomposition of the mesoscale tracer flux $\mathbf{F}_\tau = \overline{\mathbf{U}'\tau'}$ which is achieved by means of transforming mesoscale fields to isopycnal coordinates (IC) where mesoscale dynamics has the simplest form. The transformation results in splitting $\mathbf{F}_\tau$ into two components $\tilde{\mathbf{F}}_b$ and $\tilde{\mathbf{F}}_\tau$: the former is determined by buoyancy mesoscale dynamics only and has a trivial kinematic dependence on the mean tracer field, the latter is determined by mesoscale tracer dynamics. Thus, the problem of modelling (parameterizing) $\mathbf{F}_\tau$ in ZC is divided in two stages which can be termed kinematic and dynamic. The kinematic stage consists in adequate decomposing $\mathbf{F}_\tau$, and the result is expressed in terms of mesoscale fields. The dynamic stage consists in applying a specific dynamic mesoscale model to parameterize the components of $\mathbf{F}_\tau$. In this article, we show that some components of $\mathbf{F}_\tau$ are missing in ZC-OGCMs tracer equation and that their contribution is of the same order of magnitude as the mesoscale contribution itself. We also show that $\mathbf{F}_\tau$ has components across **mean** isopycnals and that their existence is consistent with the adiabatic approximation which requires vanishing all fluxes across isopycnal surfaces. As for practical results, we derive the complete equation for the large scale tracer in ZC-OGCMs and present the parameterization of the terms which have been missing thus far.

Keywords: Diapycnal mesoscale fluxes; Adiabatic approximation; Diffusion tensor; Dynamic mesoscale model

1. Introduction

As McDougall and McIntosh (2001, hereafter MM01) showed, the tracer Eulerian-averaged equation (EAE) in z-coordinates (ZC) used in OGCMs, is incomplete due to the incomplete representation of the adiabatic mesoscale mixing. Perhaps, this

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incompleteness was one of the motives to develop an alternative approach which is the time-residual-mean (TRM) model whose tracer equation is complete. The latter looks similar to the incomplete EAE used thus far in ZC-OGSMs. On this basis some physical oceanographers conclude that solutions of the incomplete EAE may be interpreted as that of TRM formalism provided one interprets solutions for Eulerian mean tracer field as that for thickness-weighted one. However, as we discuss in Conclusion section, this is not the case. Although TRM may become a good alternative of EAE, at present there exists a plenty of EAE codes while three-dimensional TRM codes for climate modelling simulations are not available yet (although in the framework of TRM formalism there were performed two-dimensional numerical experiments by Aiki *et al.* (2004) for a zonally uniform channel). In addition, in the mixed layer (ML) the TRM scheme is not applicable as well as the isopycnal coordinates (IC) formalism. Applying TRM in the ocean interior and EAE in ML, one will be confronted with the problem of matching Eulerian and thickness-weighted mean fields and mesoscale fluxes at the boundary between the upper layer including ML and the transition layer, and the interior. Thus, a work on improving ZC-OGCMs makes sense.

Returning to the missing terms in EAE for tracer fields, we recall that MM01 presented them in the second order in fluctuating fields (see MM01 equations (54), (55)) and noticed that these terms 'would be very difficult to parameterize'. In this study we show that the problem can be solved with the use of an optimal decomposition of the mesoscale tracer flux

$$\mathbf{F}_\tau = \overline{\mathbf{U}'\tau'} \tag{1a}$$

in EAE for tracer τ

$$\frac{\partial\bar{\tau}}{\partial t} + \bar{\mathbf{U}}\cdot\nabla\bar{\tau} + \nabla\cdot\mathbf{F}_\tau = Q_\tau, \tag{1b}$$

where $\mathbf{U} = \bar{\mathbf{U}} + \mathbf{U}'$ is the 3D-velocity field (mean and fluctuating correspondingly), Q_τ is a diabatic source (sink) of the tracer τ. In the present article we concentrate ourselves in the adiabatic (reversible) contribution of mesoscale eddies to the flux $\mathbf{F}_\tau$. The optimal decomposition of $\mathbf{F}_\tau$ from viewpoint of modelling is determined by the fact that appropriate coordinates for mesoscale modelling are IC where mesoscale treatment is more transparent than in ZC. This is true not only for dynamic mesoscale models like that developed by the authors recently (Canuto and Dubovikov 2005, 2006, Dubovikov and Canuto 2005; hereafter OM1, OM2, GAFD) but also for a phenomenological approach like GM (Gent and McWilliams 1990, Gent *et al.* 1995). The reason is that in IC, in the adiabatic approximation, the ocean flow occurs along isopycnal surfaces and can be considered as a set of 2D isopycnal turbulent flows between which non-linear interactions are negligible. As a result, to parameterize mesoscale fluxes in ZC, one needs to begin with expressing eddy fields in equation (1a) in terms of that in IC. The transformation process is far from trivial due to the random nature of the density field. Nevertheless, as shown in OM2 and GAFD, the transformation relations can be expanded in powers of the small parameter $|h''/\bar{h}| < 0.1$ where $h = z_\rho$ is the thickness of isopycnal layers and $''$ denote mesoscale fields in IC. In this article, we restrict ourselves with the lowest-order approximation in which, as discussed in Appendix A of OM2 (notice that the notations $'$ and $''$ in OM2 and below do not coincide), we have (see also MM01, equations (28), (39) and comments above the latter formula which point out that fluctuating fields in IC in both the thickness-weighted form and non-thickness-weighted one coincide at leading order considered in MM01)

$$\mathbf{U}' = \mathbf{U}'' - N^{-2}\bar{\mathbf{U}}_z b', \qquad \tau'' = \tau' - N^{-2}\bar{\tau}_z b', \tag{1c}$$

where b is the buoyancy field, N is the Brunt-Vaisala frequency, $N^2 = \bar{b}_z$. As discussed in OM2 and GAFD, the first term on the right-hand side of the first relation (1c) exceeds the second one, at least, in order of magnitude and we neglect it (this conclusion follows from the simple fact that the mesoscale velocity exceeds the mean one while for all other fields, analogous inequalities are opposite). In fact, as we discussed in OM2 and GAFD, the typical value of the field $(N^{-2}b')^2 \approx (z')^2$ is $\sim 10^3\,\mathrm{m}^2$ due to the filling factor of mesoscale eddies ~ 0.1. Next, taking into account the typical values $|\mathbf{U}_z| \sim 10^{-4}\,\mathrm{s}^{-1}$, $N^2 \sim 10^{-5}\,\mathrm{s}^{-2}$ and $U' \sim 0.1\,\mathrm{ms}^{-1}$, we conclude that the first term on the right-hand side of first relation (1c) exceeds the second one, at least, in order of magnitude. Thus, in accordance with the discussed arguments, we decompose $\mathbf{F}_\tau$ as follows:

$$\mathbf{F}_\tau = \tilde{\mathbf{F}}_\tau + \tilde{\mathbf{F}}_b, \tag{1d}$$

where

$$\tilde{\mathbf{F}}_\tau = \overline{\mathbf{U}'\tau''} \approx \overline{\mathbf{U}''\tau''}, \qquad \tilde{\mathbf{F}}_b = N^{-2}\bar{\tau}_z \mathbf{F}_b = \left(\frac{\partial\bar{\tau}}{\partial\bar{b}}\right)\mathbf{F}_b \tag{1e}$$

and $\mathbf{F}_b = \overline{\mathbf{U}'b'}$ is the buoyancy flux. Substituting decomposition (1d) in equatin (1b), we notice that the mesoscale-induced mixing of $\bar{\tau}$ is effected by the two dynamically different fluxes $\tilde{\mathbf{F}}_b$ and $\tilde{\mathbf{F}}_\tau$. The former has a trivial kinematic dependence on $\bar{\tau}$ and does not depend on mesoscale tracer dynamics. Thus, the corresponding mixing of the mean tracer can be termed 'kinematical' in respect of $\bar{\tau}$. The parameterization of $\tilde{\mathbf{F}}_b$ may be obtained without developing tracer dynamics once we have studied buoyancy mesoscale dynamics in our previous work. In contrast with $\tilde{\mathbf{F}}_b$, the flux $\tilde{\mathbf{F}}_\tau$ is formed by mesoscale tracer dynamics and so the corresponding mixing may be termed dynamic. Further decompositions of the fluxes $\tilde{\mathbf{F}}_b$ and $\tilde{\mathbf{F}}_\tau$ are determined by the criterion of simplicity of the parameterization and interpretation of their components as well as by the condition that (1b) to be formally close to the tracer equation used at present in OGCMs

$$\frac{\partial\bar{\tau}}{\partial t} + (\bar{\mathbf{U}} + \mathbf{u}_M)\cdot\nabla\bar{\tau} = D_R + Q_\tau, \tag{2a}$$

where $\mathbf{u}_M(\mathbf{u}^+, w^+)$ is the eddy induced velocity and D_R is the diffusion term. Usually in OGCMs $\mathbf{u}_M(\mathbf{u}^+, w^+)$ is parameterized within the GM model (Gent and McWilliams 1990, Gent *et al.* 1995), while D_R is parameterized in the Redi (1982) form

$$\mathbf{u}^+_{GM} = -\frac{\partial}{\partial z}(\kappa_M \mathbf{L}), \qquad w^+_{GM} = \nabla_H\cdot(\kappa_M \mathbf{L}), \tag{2b}$$

$$D_R = \nabla\cdot\mathbf{K}\cdot\nabla\bar{\tau}, \qquad \mathbf{K} = \kappa_M(\delta^{(2)} + \mathbf{Lk} + \mathbf{kL} + \mathbf{L}\cdot\mathbf{Lkk}), \tag{2c}$$

where $\delta^{(2)}$ is the 2D Kroneker tensors, $\mathbf{L} = -N^{-2}\nabla_H \bar{b}$ is the slope of mean isopycnals, ∇_H is the horizontal gradient operator, $\mathbf{k}$ is the unit vertical vector. Usually the diffusivity κ_M is adopted to be constant $\sim 10^3\,\mathrm{m^2 s^{-1}}$ although in reality it is variable.

The discussion above (2a) implies that the problem of modelling (parameterizing) the mesoscale tracer flux in ZC is divided in the two stages which can be termed 'kinematic' and dynamic. The approaches to kinematic and dynamic problems are quite different. In fact, the first one is model-independent and its solution is exact and of general interest while the second one is specific for an adopted dynamic approximation and, as a rule, is more disputable. So, it is reasonable to split the analysis into two parts. In the present work we show that OGCMs EAE for tracer misses some components of $\mathbf{F}_\tau$ which are of the same order of magnitude as the mesoscale contribution itself and radically influence upon the mesoscale diffusivity tensor and the mesoscale advection velocity. Furthermore, the flux $\mathbf{F}_\tau$ contains components across **mean** isopyenals. Such fluxes were neglected in OGCMs thus far on the basis that in the adiabatic approximation all diapycnal fluxes in IC vanish. Nevertheless, using an eddy resolving code, Gille and Davies (1999) found that the eddy induced diapycnal flux of the buoyancy field Σ^b 'is too large to be ignored'. Tandon and Garrett (1996), Radko and Marshall (2004a,b) and Eden *et al*. (2005) from different arguments also concluded that Σ^b is not zero. In the previous work OM2, GAFD we have derived the expressions for Σ^b in terms of large-scale fields in the framework of the mesoscale dynamical model and have concluded that the association with Σ^b diffusivity achieves a maximum where the eddy potential energy is maximal and there it can exceed the diabatic (background) diffusivity $\sim 0.1\,\mathrm{cm^2\,s^{-1}}$. The depth of the maxima is about 600 m which is below the mixed layer. This conclusion is confirmed by the recent simulations by Henning and Vallis (2005). The consensus is therefore that Σ^b cannot be ignored. We discuss the problem of consistency of fluxes across mean isopycnals with the adiabatic approximation and show that the problem ensues from the incompatibility of the definitions of the diapycnal fluxes and the diapycnal velocity in ZC and IC: in the latter case, the diapycnal velocity is defined with respect to the isopycnal surfaces which vary in time while in ZC the diapycnal velocity is computed with respect to mean isopycnals at a given moment, i.e. frozen isopycnals. Generalizing the first definition of diapycnal velocity to an arbitrary coordinate system, we show: (a) the flux of any field across isopycnals has two parts: the first represents the flux across frozen (stationary) isopycnals and the second represents the flux due to the time variation of the isopycnal surfaces themselves; (b) in the adiabatic approximation the sum of the two fluxes is indeed zero while each of the two components can be non-zero; and (c) a flux across mean isopycnals is not the full flux and may be non-zero; in particular, Σ^b equals the buoyancy flux across the frozen isopycnals; in addition, it coincides with the opposite sign with the rate of the potential energy production and therefore it is non-zero and negative. Therefore the residual mesoscale diffusivity

$$\kappa_m^r \equiv -N^{-2}\Sigma^b, \tag{3a}$$

which was introduced in OM2, is positive and represents a down-gradient diffusion across mean isopycnals. The parameterization of κ_m^r in terms of large scale fields has been derived in OM2 in the framework of the dynamical mesoscale model. We stress that although this diffusivity is diapycnal, it represents an adiabatic mesoscale mixing. Thus, one can conclude that the adiabatic mesoscale mixing is represented incompletely in the tracer OGCM equation in ZC which, therefore, should be supplemented, at the very least, by the divergence of the residual diapycnal tracer flux

$$\Sigma_{\tau}^{r} = -\kappa_{m}^{r}\bar{\tau}_{z} \tag{3b}$$

which is a counterpart of the flux Σ^{b}. In this study we also find other components of $\mathbf{F}_{\tau}$ which are missing in OGCMs. As for practical results, we derive the complete EAE for the large-scale tracer in ZC-OGCMs and present the parameterization of the mesoscale tracer fluxes $\tilde{\mathbf{F}}_{\tau}$ and $\tilde{\mathbf{F}}_{b}$ (1e).

2. Decomposition of $\tilde{F}_b$ and $\tilde{F}_\tau$ and complete mean tracer equation in ZC

We begin with decomposing the flux $\tilde{\mathbf{F}}_{\tau}$ and impose the condition that one of its components yields a diffusion term analogous to the Redi form (2c). This condition is satisfied if we decompose $\tilde{\mathbf{F}}_{\tau}$ into isopycnal and diapycnal components $\tilde{\mathbf{F}}_{\tau}^{i}$ and $\tilde{\Sigma}^{\tau}$. We obtain

$$\tilde{\Sigma}^{\tau} = N^{-2}\tilde{\mathbf{F}}_{\tau}\cdot\nabla\bar{b}, \quad \tilde{\mathbf{F}}_{\tau}^{i} = \tilde{\mathbf{F}}_{\tau} - \tilde{\Sigma}^{\tau}\mathbf{n}_{\bar{b}}, \tag{4a}$$

where $\mathbf{n}_{\bar{b}}$ is the unit vector normal to mean isopycnals. With account for the first order of the small parameter $|\bar{b}_{z}^{-1}\nabla_{H}\bar{b}|$ we have

$$\mathbf{n}_{\bar{b}} = \mathbf{k} + \bar{b}_{z}^{-1}\nabla_{H}\bar{b}. \tag{4b}$$

To simplify the further derivations, we remember that finally we will compute the divergence of all fluxes. Since the vertical components of all diapycnal fluxes considerably exceed the horizontal ones and, in addition, all vertical derivatives considerably exceed horizontal ones, we may keep only the term $\mathbf{k}$ in relation (4b) when substituting it into (4a). Then we derive

$$\tilde{\mathbf{F}}_{\tau}^{i} = \tilde{\mathbf{F}}_{\tau}^{H} - \Sigma_{H}\mathbf{k}, \qquad \tilde{\mathbf{F}}_{\tau}^{H} = \overline{\mathbf{u}'\tau''} \approx \overline{\mathbf{u}''\tau''}, \qquad \Sigma_{H} = -\tilde{\mathbf{F}}_{\tau}^{H}\cdot\mathbf{L}, \tag{4c}$$

where $\mathbf{u}$ is the horizontal component of the velocity field $\mathbf{U}$. To obtain the contribution of $\tilde{\mathbf{F}}_{\tau}$ to (1b), one needs to apply the operator ∇ to the isopycnal and diapycnal components of $\tilde{\mathbf{F}}_{\tau}$ given in (4a,c). We obtain the following result:

$$\nabla\cdot\tilde{\mathbf{F}}_{\tau} = \nabla\cdot\tilde{\mathbf{F}}_{\tau}^{i} + \tilde{\Sigma}_{z}^{\tau}, \qquad \nabla\cdot\tilde{\mathbf{F}}_{\tau}^{i} = \nabla_{H}\cdot\tilde{\mathbf{F}}_{\tau}^{H} - \frac{\partial\Sigma_{H}}{\partial z}. \tag{5a}$$

The second expression represents a diffusion along mean isopycnals. In fact, with account for (4c), it can be rewritten in the following form:

$$-D(\tau) = \nabla\cdot\tilde{\mathbf{F}}_{\tau}^{i} = N^{2}\nabla_{\bar{\rho}}\cdot(N^{-2}\tilde{\mathbf{F}}_{\tau}^{H}), \tag{5b}$$

where

$$\nabla_{\bar{\rho}} = \nabla_H + \mathbf{L}\frac{\partial}{\partial z} \tag{5c}$$

is the gradient operator in a system where horizontal coordinate surfaces are the mean isopycnals. Next, since in the considered approximation $\nabla_\rho = \nabla_{\bar{\rho}}$, with account for the second relation (4c), the right-hand side of (5b) coincides with the diffusion term in IC given in (5) of Gent *et al.* (1995), whose parameterization was suggested in their equation (6). In the considered approximation in which $\bar{\tau}$ and $\bar{\bar{\tau}}$ differ only at the second order in fluctuating fields (see MM1 comments above equation (39)), the suggested parameterization can be presented in the following form

$$\tilde{\mathbf{F}}_\tau^H = -\kappa_M \nabla_\rho \bar{\bar{\tau}} = -\kappa_M \nabla_{\bar{\rho}} \bar{\tau}, \tag{5d}$$

where $\bar{\bar{\tau}}$ is the mean tracer in IC. In OGCMs it is adopted $\kappa_M = \text{const.} \sim 10^3\,\text{m}^2\,\text{s}^{-1}$. Then the last expression (5d) together with (5b) yields the Redi diffusion (2c). Thus, from equations (5a,b) it follows

$$\nabla \cdot \tilde{\mathbf{F}}_\tau = -D(\tau) + \tilde{\Sigma}_z^\tau. \tag{5e}$$

Next, we decompose the flux $\tilde{\mathbf{F}}_b$ requiring that after substituting into (1b), one of its components reproduces the term of the OGCMs tracer equation (2a) with the eddy-induced velocity $\mathbf{u}_M$. The appropriate decomposition is into components $\tilde{\mathbf{F}}_b^\tau$ and $\tilde{\Sigma}^b$ which are the projections of the flux $\tilde{\mathbf{F}}_b$ on a surface of constant $\bar{\tau}$ and on the normal to it correspondingly. Thus,

$$\tilde{\Sigma}^b = \bar{\tau}_z^{-1} \tilde{\mathbf{F}}_b \cdot \nabla \bar{\tau} = N^{-2} \mathbf{F}_b \cdot \nabla \bar{\tau}. \tag{6a}$$

The normal $\mathbf{n}_{\bar{\tau}}$ may be presented analogously to (4b) with the substitution $\bar{b} \to \bar{\tau}$. Repeating the arguments which are below (4b), we may substitute $\mathbf{n}_{\bar{\tau}} \to \mathbf{k}$. Then from (6a) we obtain

$$\tilde{\mathbf{F}}_b^\tau = N^{-2}\bar{\tau}_z \mathbf{F}_b - \tilde{\Sigma}^b \mathbf{k} = N^{-2}\bar{\tau}_z \mathbf{F}_b^H - N^{-2}(\nabla_H \bar{\tau}) \cdot \mathbf{F}_b^H \mathbf{k}, \tag{6b}$$

where $\mathbf{F}_b^H = \overline{\mathbf{u}'b'}$ is the horizontal buoyancy flux. Taking the divergences of $\tilde{\mathbf{F}}_b$ and accounting for the well-known relations

$$\mathbf{u}^+ = -\frac{\partial}{\partial z}(N^{-2}\mathbf{F}_b^H), \qquad w^+ = \nabla_H \cdot (N^{-2}\mathbf{F}_b^H), \tag{6c}$$

we obtain

$$\nabla \cdot \tilde{\mathbf{F}}_b = \nabla \cdot \tilde{\mathbf{F}}_b^\tau + \tilde{\Sigma}_z^b, \qquad \nabla \cdot \tilde{\mathbf{F}}_b^\tau = \mathbf{u}_M \cdot \nabla \bar{\tau}. \tag{6d}$$

Substituting (5e), (6d), and (1d) into (1b) we obtain the complete mean tracer equation in ZC

$$\frac{\partial \bar{\tau}}{\partial t} + (\mathbf{U} + \mathbf{u}_M) \cdot \nabla \bar{\tau} = D(\tau) - \Sigma^{\tau}_z + Q_{\tau}, \tag{7a}$$

where

$$\Sigma^{\tau} = \tilde{\Sigma}^{\tau} + \tilde{\Sigma}^{b}. \tag{7b}$$

Recall that the term with $\mathbf{u}_M$ in (7a) represents the contribution of the component of $\tilde{\mathbf{F}}_b$ within a surface of constant $\bar{\tau}$ but not of constant $\bar{b}$ as in the case of the flux $\mathbf{F}_b$ in the buoyancy equation (Andrews and McIntyre 1976). In the same time, the term Σ^{τ} on the right-hand side of (7a) is composed by the diapycnal flux $\tilde{\Sigma}^{\tau}$ (4a) and by the flux $\tilde{\Sigma}^{b}$ (6a) across a surface of constant $\bar{\tau}$, as given in (7b). Recall that all mesoscale terms in (7a) with the exception of Q_{τ}, represent the eddy adiabatic mixing. Notice that the OGCMs tracer equation (2a) includes only the isopycnal component of the flux $\tilde{\mathbf{F}}_{\tau}$ and the component within a surface of constant $\bar{\tau}$ of the flux $\tilde{\mathbf{F}}_b$ but lacks the orthogonal to the components which yield the mesoscale term Σ^{τ}_z in the complete equation (7a). As we show below, the missing term is of the same order of magnitude as the mesoscale term $\mathbf{u}_M \cdot \nabla \bar{\tau}$ and therefore has to be parameterized and accounted for. Equation (7a) is applicable equally to passive and active tracers, like T and S, and even to the buoyancy (density) field. In the latter case, we have

$$D(\tau) = 0, \qquad \tilde{\Sigma}^{\tau} = 0, \qquad \tilde{\Sigma}^{b} = \Sigma^{b} \equiv N^{-2} \mathbf{F}_b \cdot \nabla \bar{b} \tag{7c}$$

and thus (7a) reduces to the well-known mean buoyancy equation (see, for example, Treguier *et al.* 1997, equation (5))

$$\frac{\partial \bar{b}}{\partial t} + (\mathbf{U} + \mathbf{u}_M) \cdot \nabla \bar{b} = -\Sigma^{b}_z + Q_b. \tag{7d}$$

Finally, we discuss the boundary conditions for the fluxes Σ^{τ}, $\tilde{\Sigma}^{\tau}$, and $\tilde{\Sigma}^{b}$ at the surface and bottom. To this end we recall that the mesoscale fields z'' and $b' = -z'' \bar{b}_z$ vanish at that boundaries (see OM1) and therefore this is true also for the buoyancy flux $\mathbf{F}_b = \overline{\mathbf{U}' b'}$. Then from definitions (6a,c) it follows that

$$w^{+}(z = 0, -H) = \tilde{\Sigma}^{b}(z = 0, -H) = 0. \tag{7e}$$

As far as the flux $\tilde{\Sigma}^{\tau}$, next we derive the representation (12d) for it from which it follows that $\tilde{\Sigma}^{\tau}$ satisfies the same condition as Σ^{τ}:

$$\tilde{\Sigma}^{\tau}(z = 0, -H) = \Sigma^{\tau}(z = 0, -H) = 0. \tag{7f}$$

To make (7a) usable in OGCMs, one needs to parameterize the mesoscale functions $\mathbf{u}_M$, $D(\tau)$, Σ^{τ} in terms of large-scale fields that we discuss in section 5. In the next two sections we discuss the problem of consistency of the tracer fluxes across mean isopycnals, like Σ_τ^r, (3b), or $\tilde{\Sigma}^\tau$, (4a), with the adiabatic approximation since it is frequently stated in the literature that they are inconsistent. For example, Gille and Devis (1999) who were first to prove numerically that Σ^b is not zero, interpreted their result as follows: 'We find that eddy-induced advection, which is adiabatic in the high-resolution model, will appear diabatic if we examine only components resolved by coarse-resolution models.' Another example: Eden *et al.* (2005) have also studied the problem of Σ^b. While recognizing that it 'may be important in the dynamics of the large-scale ocean circulation', they state that 'it may not be physically justified' and that 'desirable feature is that there should be no diapycnal fluxes in the mean equations.' On this premises, they exploit the gauge freedom to add a curl to the eddy fluxes so as to make Σ^b disappear from the buoyancy equation. If the elimination of Σ^b were dictated, say, by numerical reasons, there could be no objections. However, the authors express their motivation in the statement (i) of section 1 that the existence of diapycnal fluxes is not consistent with the adiabatic approximation. Thus, the purpose of the next two sections is to show that this statement is incorrect. In addition, this analysis is helpful for parameterizing the fluxes $\tilde{\Sigma}^\tau$ and $\tilde{\Sigma}^b$.

3. Diapycnal fluxes and the adiabatic approximation

As we just show, the problem ensues from the incompatibility of the traditional definitions of the diapycnal fluxes and the diapycnal velocity in ZC and IC. In fact, the flux Σ^b (7c) is traditionally termed the diapycnal buoyancy flux in ZC. This implies that the diapycnal velocity must be defined with respect to mean isopycnals taken at a given moment, i.e. isopycnals frozen in time

$$w_d = \mathbf{U} \cdot \mathbf{n}_{\bar{b}}, \tag{8a}$$

where $\mathbf{n}_{\bar{b}}$ is the unit vector normal to a mean isopycnal given in (4b). Indeed, substituting (8a) and (4b) in the definition of the diapycnal buoyancy flux $\overline{b' w_d} = \Sigma^b$, we obtain expression (7c) for Σ^b. On the other hand, in IC the diapycnal velocity is defined with respect to isopycnal coordinate surfaces which vary in time in physical space. One can present this definition in an arbitrary coordinate system as follows:

$$w_D = (\mathbf{U} - \mathbf{U}_s) \cdot \mathbf{n}_b, \tag{8b}$$

where $\mathbf{U}_s$ is the velocity of a given element of an isopycnal surface and $\mathbf{n}_b$ is the unit vector normal to this element. We use the different subscripts in the designations of the diapycnal velocities (8a,b) to distinguish their definitions with respect to frozen and varying isopycnals correspondingly. Since both variables w_d and w_D are useful in different analyses, we suggest to term them differently keeping the term diapycnal velocity for w_d and using the term *diapycnic* velocity for w_D in the spirit of the Bleck's (2002) terminology for isopycnal coordinates. In these terms, the adiabatic approximation implies vanishing the diapycnic velocity, i.e.

$$w_D = 0. \tag{8c}$$

Thus, for analysis of the adiabatic approximation, the appropriate flux of a field A is the diapycnic one

$$F_A = \overline{w_D A} = \overline{(\mathbf{U} - \mathbf{U}_s)\cdot\mathbf{n}_b A}, \tag{9a}$$

which is the sum of the two fluxes

$$F_A^f = \overline{\mathbf{U}\cdot\mathbf{n}_b A}, \quad F_A^s = -\overline{\mathbf{U}_s\cdot\mathbf{n}_b A}, \tag{9b}$$

where F_A^f is the flux across the frozen isopycnal and F_A^s is the contribution of the isopycnal variation to the diapycnic flux F_A. Thus (9a) can be written as

$$F_A = F_A^f + F_A^s = \overline{w_D A}. \tag{9c}$$

Therefore, under condition (8c), we have

$$F_A = F_A^f + F_A^s = 0 \tag{9d}$$

i.e. in the adiabatic approximation, all diapycnic fluxes vanish, but this is not the case for diapycnal fluxes.

Finally, for using in concrete analysis, we express the fluxes F_A^f and F_A^s in IC and ZC. In the latter case, analogously to (4b), we have

$$\mathbf{n}_b = \mathbf{k} + b_z^{-1}\nabla_H b, \quad \mathbf{U}_s = -b_z^{-1} b_t \mathbf{k}. \tag{10a}$$

Hence it follows

$$\mathbf{U}\cdot\mathbf{n}_b = w + b_z^{-1}\mathbf{u}\cdot\nabla_H b, \quad \mathbf{U}_s\cdot\mathbf{n}_b = -b_z^{-1} b_t. \tag{10b}$$

Substituting this result into (9b), the two fluxes become

$$F_A^f = \overline{Aw} + \overline{Ab_z^{-1}\mathbf{u}\cdot\nabla_H b}, \quad F_A^s = \overline{Ab_z^{-1} b_t}. \tag{10c}$$

Transformation of this result to IC yields

$$F_A^f = \overline{Aw} - \overline{A\mathbf{u}\cdot\nabla_\rho z}, \quad F_A^s = -\overline{Az_t}. \tag{10d}$$

To consider the problem of consistency of fluxes across mean isopycnals with the adiabatic approximation, one needs to express such fluxes in terms of F_A^f and F_A^s that is helpful also for parameterizing the fluxes. In particular, we are interested in the diapycnal fluxes Σ^b and $\tilde{\Sigma}^\tau$ and their relation to the adiabatic approximation.

4. Expressions for Σ^b and $\tilde{\Sigma}^\tau$ in terms of F^f_A and F^s_A

We begin from the buoyancy flux Σ^b (7c). To study the problem, we choose $A = B \equiv N^{-2} b_z b' \approx b'$. Then from (10c) and (9d) we obtain correspondingly

$$N^2 F^s_B = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\overline{b'^2}\right), \qquad N^2 F^f_B = N^2 \Sigma^b + \bar{\mathbf{U}}\cdot\nabla\left(\tfrac{1}{2}\overline{b'^2}\right) + \tfrac{1}{2}\overline{\mathbf{U}'\cdot\nabla b'^2}, \tag{11a}$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\overline{b'^2}\right) + \bar{\mathbf{U}}\cdot\nabla\left(\tfrac{1}{2}\overline{b'^2}\right) + \tfrac{1}{2}\overline{\mathbf{U}'\cdot\nabla b'^2} = -N^2\Sigma^b. \tag{11b}$$

The second and third terms on the left-hand side of (11b) represent the advection and diffusion of the buoyancy variance whereas the right-hand side represents its production. Following MM01, we argue that the diffusion term is negligible in comparison with the advection one since the former is a third-order term in fluctuating fields. As shown in GAFD, the advection term, in turn, is negligible in comparison with the production. Neglecting the second and third terms on the left-hand side of (11b) and the same terms in (11a), we obtain the final result:

$$-F^s_B = F^f_B = \Sigma^b, \tag{11c}$$

which shows that Σ^b is not the full diapycnic buoyancy flux, rather it relates to the components either F^f_B or $-F^s_B$ of the diapycnic flux of the field $B \equiv N^{-2} b_z b' \approx b'$.

Notice that

$$W = \tfrac{1}{2} N^{-2}\overline{b'^2}, \tag{11d}$$

where W is the eddy potential energy. Accounting for that the characteristic time-scale of the variation of N is longer than that of W and neglecting the advection and diffusion terms in (11b), we obtain the equation

$$\frac{\partial W}{\partial t} = -\Sigma^b, \tag{11e}$$

which implies that the production of eddy potential energy is local, i.e., the growth rate of the potential energy is determined by its production in the same place. Result (11e) may be interpreted as the condition of the adiabatic approximation instead of the previously adopted condition $\Sigma^b = 0$.

Notice that it is usually suggested (see, for example, Treguier *et al.* 1997) to apply (11b) to a stationary flow. However, such an analysis is not applicable in the adiabatic approximation. In fact, in the case of stationary flows in the complete equation for $\partial\overline{b'^2}/\partial t$ which contains also the diabatic term, the production term $-2N^2\Sigma^b$ is mainly balanced by the dissipation due to the diabatic mixing. In the adiabatic approximation the dissipation is absent. Then, apart from the negligible diffusion and advection terms, the production term in (11b) may be balanced only by the adiabatic growth rate of the buoyancy variance which because of $\overline{b'^2} = 2N^2 W$, is proportional to the adiabatic growth rate of eddy potential energy W_t.

As we discussed in Introduction, the component Σ^r_τ of the diapycnal tracer flux is generated by transport of the mean tracer by the diapycnal mesoscale buoyancy flux and both fluxes have the same diffusivity κ^r_m, (3a,b), whereas the diapycnal tracer flux $\tilde{\Sigma}^\tau$, (4a), is determined by tracer dynamics. To parameterize this flux, one should analyze it in IC. So we consider the status of $\tilde{\Sigma}^\tau$ in terms of $F^f_A$and F^s_A in IC using relations (10d). Choosing $A = \tau''$ we obtain

$$F^s_{\tau''} = -\overline{\tau'' z''_t}, \qquad F^f_{\tau''} = \overline{\tau'' w''} - \overline{\tau'' \mathbf{u}''} \cdot \nabla_\rho \bar{z} - \overline{\tau'' \nabla_\rho z''} \cdot \bar{\mathbf{u}} - \overline{\tau'' \mathbf{u}'' \cdot \nabla_\rho z''}. \tag{12a}$$

To relate the second expression to $\tilde{\Sigma}^\tau$, (4a), we transform the latter to IC. In the main order in the small parameter $h''/\bar{h}$ we use (1c) and the relation $\nabla_H \bar{b} = -N^2 \nabla_\rho \bar{z}$. Retaining only the first term on the right-hand side of the first relation (1c), from (4a) we obtain

$$\tilde{\Sigma}^\tau = \overline{\tau'' w''} - \overline{\tau'' \mathbf{u}''} \cdot \nabla_\rho \bar{z}. \tag{12b}$$

Neglecting the term of the third order in fluctuating fields in (12a) and then comparing expressions (12a,b), we deduce

$$\tilde{\Sigma}^\tau = F^f_{\tau''} + \overline{\tau'' \nabla_\rho z''} \cdot \bar{\mathbf{u}}. \tag{12c}$$

Finally, from (9d), (12c), and the first equation (12a) we obtain the following relation

$$\tilde{\Sigma}^\tau = \overline{\tau''(z''_t + \bar{\mathbf{u}} \cdot \nabla_\rho z'')}, \tag{12d}$$

which will be used for parameterizing $\tilde{\Sigma}^\tau$ in the next section. Recall that in (12) averaging is performed in IC. Results (12c,d) would be analogous to one (11c,d) if we did not neglect the advection term in (11b) (the second term) which is analogous to the last terms in (12c,d). In the next section we show that the first term on the right-hand side of (12c) typically dominates over the second one although the latter is not negligible and should be taken into account.

5. Mesoscale parameterization in complete tracer equation (7a)

As we discussed after (1e), the mesoscale tracer flux is split into ones $\tilde{\mathbf{F}}_b$ and $\tilde{\mathbf{F}}_\tau$ which are determined correspondingly by buoyancy and tracer mesoscale dynamics. Therefore, in order to parameterize the functions $\mathbf{u}_M$ and $\tilde{\Sigma}^b$ originating from $\tilde{\mathbf{F}}_b$ and given in equation (6c,a), it is sufficient to solve only buoyancy (density) equation of the dynamic mesoscale model what we did in the previous work (OM1, 2, GAFD). Thus, the parameterizations of these functions either were derived there as for the case of $\mathbf{u}_M$, or can easily be obtained from the previous results. In fact, to parameterize $\tilde{\Sigma}^b$, it is convenient to decompose it as follows:

$$\tilde{\Sigma}^b = \Sigma^r_\tau + \tilde{\Sigma}^r_\tau, \tag{13a}$$

where Σ^r_τ is given by (3a,b), (7c) and

$$\tilde{\Sigma}^r_\tau = N^{-2}\mathbf{F}^H_b \cdot \nabla_{\bar{\rho}}\bar{\tau}. \tag{13b}$$

In (3b) the flux Σ^r_τ is expressed through the residual diapycnal diffusivity $\kappa^r_m = -N^{-2}\Sigma^b$ which was parameterized and evaluated in the previous work (OM2, AGFD2). In particular, as shown in OM2, κ^r_m achieves a maximum where the eddy potential energy is maximal (at about 600 m depth) and there it can exceed the diabatic (background) diffusivity. Thus account for the term Σ^r_τ in the tracer equation (7a,b) is important.

Next, $\tilde{\Sigma}^r_\tau$ (13b) is expressed through the flux $\mathbf{F}^H_b$ which relates to $\mathbf{u}_M$ as given in (6c). Since the dynamic model yields the boundary condition $b' \approx 0$ at the surface and bottom, we deduce the analogous condition for the buoyancy flux. Therefore

$$\mathbf{F}^H_b = -N^2 \int_{-H}^{z} \mathbf{u}^+(\zeta)\mathrm{d}\zeta. \tag{13c}$$

From this relation and equation (13b) it follows that the contribution to the tracer equation (7a,b) of the eddy induced advection and of $\partial\tilde{\Sigma}^r_\tau/\partial z$ are of the same order of magnitude. Therefore, account for the latter is important.

Contrary to the functions $\mathbf{u}_M$, Σ^r_τ and $\tilde{\Sigma}^r_\tau$, in order to parameterize the functions $D(\tau)$ and $\tilde{\Sigma}^\tau$ which originate from $\tilde{\mathbf{F}}_\tau$ and are determined by mesoscale tracer dynamics, we need not only dynamic equations for the mesoscale velocity and thickness fields but also that for the tracer which we did not consider in the previous publications. Now this work is in progress although not finished. Below we parameterize the flux $\tilde{\Sigma}^\tau$ phenomenologically once it is commonly adopted to the phenomenological parameterization of the diffusion $D(\tau)$ in the form of (5b,d). As we discussed in Introduction, the parameterization process must begin with transforming the definition of $\tilde{\Sigma}^\tau$ given in (4a), to IC where modelling is much simpler than in ZC. The result of the transformation has been obtained in section 4 and presented in (12d). If we apply the Fourier transformation to this result both in time and space within 2D isopycnal surfaces, the expression in the parentheses in (12d) becomes $(-\mathrm{i}\omega + \mathrm{i}\bar{\mathbf{u}}\cdot\mathbf{q})z''$ where $\mathbf{q}$ is a 2D wave vector. In OM2 and GAFD we showed that for mesoscale eddies $\omega = \mathbf{q}\cdot\mathbf{u}_{dr}$ where $\mathbf{u}_{dr}$ is the eddy drift velocity whose expression in terms of large-scale fields is given in these articles. Thus, the considered Fourier transformation can be presented in the form $\mathrm{i}(\mathbf{u} - \mathbf{u}_{dr})\cdot\mathbf{q}z''$. This result allows us to present expression (12d) as follows:

$$\tilde{\Sigma}^\tau = (\bar{\mathbf{u}} - \mathbf{u}_{dr})\cdot\overline{\tau''\nabla_\rho z''}. \tag{14}$$

Therefore, the parameterization of $\tilde{\Sigma}^\tau$ reduces to that of the correlation function on the right-hand side of (14). Below we model the latter phenomenologically on the basis that the mesoscale field is quasi-geostrophic for which case we have

$$\nabla_\rho z'' = f\mathbf{k} \times \frac{\partial \mathbf{u}''}{\partial b}. \tag{15a}$$

In addition, we notice that the mesoscale fields $\mathbf{u}''$ and $\partial\mathbf{u}''/\partial b$ have approximately the same (or opposite) direction since $\mathbf{u}''$ represents a quasi-circular motion of the eddies around their centers. Using also the first relation (1c) and neglecting the second term in its right-hand side, we obtain

$$\frac{\partial \mathbf{u}''}{\partial b} = \frac{\mathbf{u}'}{u'}\frac{\partial u'}{\partial b} = \frac{\mathbf{u}'}{2}\frac{\partial(\ln\Gamma)}{\partial b} = \frac{\mathbf{u}'}{2N^2}\frac{\partial(\ln\Gamma)}{\partial z} \tag{15b}$$

where $\Gamma(z) \equiv K(z)/K_s$ is the normalized profile of eddy kinetic energy (K_s is the surface eddy kinetic energy) which was parameterized in terms of large-scale fields in OM2 and GAFD. From (14), (15) we obtain the relation

$$\tilde{\Sigma}^\tau \mathbf{k} = -\frac{f}{2N^2}(\bar{\mathbf{u}} - \mathbf{u}_{dr})\frac{\partial[\ln\Gamma(z)]}{\partial z} \times \tilde{\mathbf{F}}^H_\tau, \tag{16}$$

which reduces the parameterization of $\tilde{\Sigma}^\tau$ to modelling the flux $\tilde{\mathbf{F}}^H_\tau = \overline{\mathbf{u}'\tau''}$ whose phenomenological model is accepted in the literature in the form (5d). Thus, we obtain the result

$$\tilde{\Sigma}^\tau \mathbf{k} = -\kappa_M \mathbf{M} \times \nabla_{\bar{\rho}}\bar{\tau}, \tag{17a}$$

where

$$\mathbf{M} = -\frac{f}{2N^2}(\bar{\mathbf{u}} - \mathbf{u}_{dr})\frac{\partial[\ln\Gamma(z)]}{\partial z}. \tag{17b}$$

Finally, the contribution of $\tilde{\Sigma}^\tau$ into tracer equation (7a,b) should be compared with that of the diffusion term (5b) since the both depend on $\tilde{\mathbf{F}}^H_\tau = \overline{\mathbf{u}'\tau''}$. Evaluating ∇_ρ through the typical horizontal length scale $\sim 10^6$ m, we obtain

$$D \sim 10^{-6}\overline{\mathbf{u}'\tau''}. \tag{17c}$$

To compare D with the contribution of $\tilde{\Sigma}^\tau$, we notice that the main contribution to (16) comes from the drift velocity whose main term, as follows from equations (4e,f) of OM2, is of the order $fr_d^2|\partial\mathbf{L}/\partial z|$ where r_d is the Rossby deformation radius and $\mathbf{L}$ is the slope of isopycnals. Taking $f \sim 10^{-4}\,\mathrm{s}^{-1}$, $r_d^2 \sim 10^9\,\mathrm{m}^2$, $|\mathbf{L}| \sim 10^{-3}$, and $|\partial_z| \sim 10^{-3}\,\mathrm{m}^{-1}$, we obtain $u_{dr} \sim 10^{-1}\,\mathrm{ms}^{-1}$. Since the order of $\bar{u}$ is a few $\mathrm{cm\,s}^{-1}$, we conclude that the second terms dominate in (16). With account for this result, from (16) we obtain

$$\tilde{\Sigma}^\tau_z \sim 10^{-6}\overline{\mathbf{u}'\tau''} \sim D. \tag{17d}$$

Thus, we conclude that the term $\tilde{\Sigma}^\tau_z$ is important in the mean tracer equation (7a,b).

In conclusion, in the present section we have derived the parameterization in terms of large-scale fields for the fluxes $\tilde{\Sigma}^b$ and $\tilde{\Sigma}^\tau$ which thus far have been missing in the tracer equation of ZC-OGCMs. The parameterization is given in (13a–c), (3a,b), (17a,b) in which Σ^b, $\mathbf{u}^+$, $\Gamma(z)$, $\mathbf{u}_{dr}$ have been parameterized in OM2 (see equations (7d–g), (4a–f) and (5a–g)). In particular, in OM2 it is shown that κ_m^r vanishes at the surface and the bottom. Then from (3b) and (13b,c) it follows that the boundary condition (7e) is satisfied. The same is true for condition (7g) since the normalized profile of eddy kinetic energy $\Gamma(z)$ achieves a maximum at the surface and a local maximum at the bottom (see OM2). Thus, the derived parameterization satisfies the boundary condition (7e,g).

6. Mesoscale tracer mixing tensor

As we noticed in the end of section 2, independently of a mesoscale modelling, in (7a) the correction Σ_z^τ to the OGCMs tracer equation (2a) is of the same order as the mesoscale contribution itself. This becomes especially clear if we represent the tracer flux with use of the mixing tensor $\mathbf{J}$ (Rhines and Holland 1979, Plumb 1979, Plumb and Mahlman 1987, Griffies 1998) as

$$\mathbf{F}_\tau = -\mathbf{J}\cdot\nabla\tau \tag{18a}$$

and decompose $\mathbf{J}$ into symmetric and antisymmetric components

$$\mathbf{J} = \mathbf{K} + \mathbf{A} \tag{18b}$$

which correspond to a diffusion and advection (or skew-diffusion). The 3D advection velocity $\mathbf{U}_*$ relates to the antisymmetric tensor $\mathbf{A}$ as follows:

$$\mathbf{U}_* = -\nabla\cdot\mathbf{A}. \tag{18c}$$

Taking the divergence of this relation and taking into account the anti-symmetry of $\mathbf{A}$, we conclude that

$$\nabla\cdot\mathbf{U}_* = \mathbf{0}. \tag{18d}$$

Specifically, in the OGCMs equation (2a), $\mathbf{K}$ equals the Redi diffusion tensor given in (2c), whereas the advection velocity $\mathbf{U}_*$ equals $\mathbf{u}_M$ (2b). Next, we show that the term Σ_z^τ in equation (7a) drastically changes these relations. With this end in view, we write down the corrections to the mixing tensor and to its symmetric and antisymmetric components, as well as to the advection velocity (18c), due to the flux $\mathbf{k}\tilde{\Sigma}_\tau^r$. Straightforwardly from (13b) and (5c) we obtain the following contributions of $\tilde{\Sigma}_\tau^r$ to $\mathbf{J}$, $\mathbf{K}$, $\mathbf{A}$, and $\mathbf{U}_*$:

$$\mathbf{J}(\tilde{\Sigma}_\tau^r) = -N^{-2}\mathbf{k}\left(\mathbf{F}_b^H + \mathbf{k}\mathbf{L}\cdot\mathbf{F}_b^H\right), \tag{19a}$$

$$\mathbf{K}(\tilde{\Sigma}_\tau^r) = -\tfrac{1}{2}N^{-2}\left(\mathbf{k}\mathbf{F}_b^H + \mathbf{F}_b^H\mathbf{k}\right) - N^{-2}\mathbf{k}\mathbf{k}\mathbf{L}\cdot\mathbf{F}_b^H, \tag{19b}$$

$$\mathbf{A}(\tilde{\Sigma}^r_\tau) = -\tfrac{1}{2}N^{-2}\left(\mathbf{k}\mathbf{F}^H_b - \mathbf{F}^H_b\mathbf{k}\right), \tag{19c}$$

$$\mathbf{U}_*(\tilde{\Sigma}^r_\tau) = -\tfrac{1}{2}\mathbf{u}_M. \tag{19d}$$

The form of (19) does not depend on mesoscale modelling.

To show that the term Σ^τ_z in (7a) yields a correction to the OGCMs equation (2a) of the same order of magnitude as the mesoscale contribution, it is sufficient to demonstrate this for the GM model. Within this model, $N^{-2}\mathbf{F}^H_b = \kappa_M\mathbf{L}$ and, in addition, all fluxes across mean isopycnals, like Σ^r_τ and $\tilde{\Sigma}^\tau$, equal zero. Thus, in accordance with (7b), (13a), in equation (7a), we may substitute $\Sigma^\tau = \tilde{\Sigma}^r_\tau$ whose diffusion tensor is obtained from (19b) to be

$$\mathbf{K}^{GM}(\tilde{\Sigma}^r_\tau) = -\tfrac{1}{2}\kappa_M[(\mathbf{k}\mathbf{L} + \mathbf{L}\mathbf{k}) + 2\mathbf{L}\cdot\mathbf{L}\mathbf{k}\mathbf{k}]. \tag{20a}$$

This expression together with the Redi diffusion tensor (2c) yields

$$\mathbf{K}^{GM}_{tot} = \kappa_M\left[\delta^{(2)} + \tfrac{1}{2}(\mathbf{k}\mathbf{L} + \mathbf{L}\mathbf{k})\right]. \tag{20b}$$

In addition, with the use of (19d) we obtain

$$\mathbf{U}^{GM}_*(tot) = \tfrac{1}{2}\mathbf{u}_M \tag{20c}$$

which is only the half of the mesoscale advection velocity in the OGCMs equation (2a). The diffusivity tensor (20b) also considerably differs from the Redi one (2c) (in particular, it does not contain the vertical diffusion). Result (20c) may seem surprising since equation (7a) is equally applicable to the buoyancy in whose equation (7d) the mesoscale advection velocity is twice as (20c). The resolution of the apparent discrepancy is that the other half of the mesoscale advection term in (7d) is yielded by the diffusion (20b). In fact, from (6c) and (19b) and with account for vanishing the Redi diffusion for the buoyancy field, we deduce

$$\nabla\cdot\mathbf{K}\cdot\nabla\bar{b} = -\tfrac{1}{2}\mathbf{u}_M\cdot\nabla\bar{b}. \tag{20d}$$

On the other hand, the mesoscale contribution to the buoyancy equation may also be interpreted as the contribution of the advection velocity equal to $\mathbf{u}_M$ and of the only diffusion tensor component κ^r_m given in (3a). Since splitting the mixing tensor into symmetric and anti-symmetric parts (18b) is unique, the question arises whether deducing the mixing tensor from an expression for the divergence of the flux is unique. The answer is positive provided one imposes the additional condition that $\mathbf{J}$ does not depend on a mixed field. Indeed, under this condition, from (18a,b) we obtain

$$\nabla\cdot\mathbf{F}_\tau = -K_{ij}\frac{\partial^2\bar{\tau}}{\partial x_i\partial x_j} - \frac{\partial}{\partial x_i}\left[(K_{ij} + A_{ij})\frac{\partial\bar{\tau}}{\partial x_j}\right]. \tag{20e}$$

By equating the terms of the mean tracer equation which contains the second derivatives of $\bar{\tau}$, with the first term in (20e), one can reconstruct the symmetric tensor $\mathbf{K}$. By equating the terms of the same equation, which contain the first derivatives of $\bar{\tau}$ with the second term in (20e) and using the result for $\mathbf{K}$, one can reconstruct $\mathrm{U}_* = -\nabla \cdot \mathbf{A}$. The imposed condition is satisfied in (19)–(21) and (2c) for the tracer field and is not satisfied for the buoyancy. Thus, for buoyancy the choice of $\mathbf{J}$ on the basis of equation (7d) is not unique as we have discussed just above. However, if one defines the mixing tensor for the buoyancy $\mathbf{J}_b$ as that for a tracer with the further choice $\tau = b$, then the definition of $\mathbf{J}_b$ becomes unique. With this definition, relation (20c) is valid also for the buoyancy field in which case the other half of the mesoscale advection term in buoyancy equation (7d) is yielded by the diffusion (20b), as (20d) shows.

Finally, for completeness, we present the mixing, diffusion and skew-diffusion tensors for the fluxes Σ^r_τ and $\tilde{\Sigma}^\tau$ which are obtained straightforwardly from relations (3b) and (17a), (5c):

$$\mathbf{J}(\Sigma^r_m) = \mathbf{K}(\Sigma^r_m) = \kappa^r_m \mathbf{kk}, \qquad \mathbf{A}(\Sigma^r_m) = \mathbf{0}, \tag{21a}$$

$$\mathbf{J}(\tilde{\Sigma}^\tau) = \kappa_M(\mathbf{kk} \times \mathbf{M} + \mathbf{M} \times \mathbf{Lk}). \tag{21b}$$

Here $\mathbf{K}(\tilde{\Sigma}^\tau)$ and $\mathbf{A}(\tilde{\Sigma}^\tau)$ can be obtained by symmetrizing and anti-symmetrizing the tensor (21b).

7. Conclusion

The main goal of the present article is to find the optimal decomposition of the adiabatic component of the mesoscale tracer flux $\mathbf{F}_\tau$ in the ZC mean tracer equation from the viewpoint of its interpretation and parameterization. We show that the very fact, that IC are the most appropriate for any dynamical approach to mesoscale modelling, results in splitting the flux $\mathbf{F}_\tau$ into two components $\tilde{\mathbf{F}}_b$ and $\tilde{\mathbf{F}}_\tau$ which are of quite the different origins. The first is formed by eddy buoyancy dynamic only and depends kinematically on the **mean** tracer field (but not on eddy one) while the second depends on the mesoscale tracer field and is formed by eddy tracer dynamics (together with dynamics of other eddy fields) in IC. Correspondingly, the parameterization of $\tilde{\mathbf{F}}_b$ may be written straightforwardly on the basis of eddy buoyancy dynamics which we have studied in the previous work while eddy tracer dynamics which is necessary for parameterizing $\tilde{\mathbf{F}}_\tau$, shall be presented in the next publication. Nevertheless, in section 5 we present a phenomenological derivation of the parameterization of the full $\mathbf{F}_\tau$. The result can be used in ZC-OGCMs together with the parameterization of the mesoscale Reynolds stress obtained in OM2 and GAFD. As we have shown, the OGCMs tracer equation (2a), used thus far, lacks some important terms of the complete equation (7a). MM01 found the missing terms up to the second order in fluctuating fields in the form $-\nabla \cdot \mathbf{E}$ where $\mathbf{E}$ is some flux presented in their formula (55). To compare this result with the term $-\Sigma^\tau_z$ in the complete tracer equation (7a), we transformed the referred equation (55) with use of the evolution equations for the eddy fields entering (55). Restricting ourselves with terms of the second order in fluctuating fields we derived

$$\nabla \cdot \mathbf{E} = \Sigma_z, \tag{22a}$$

$$\Sigma = N^{-2}(\overline{\mathbf{U}'\tau'} \cdot \nabla \bar{b} + \overline{\mathbf{U}'b'} \cdot \nabla \bar{\tau}) - N^{-4}\bar{\tau}_z \overline{\mathbf{U}'b'} \cdot \nabla \bar{b}. \tag{22b}$$

This result is identical to $-\Sigma^\tau$ with account for (7b), (6a), and (4a).

Finally, we return to the issue discussed in Introduction that the incomplete OGCMs tracer equation (2a) used thus far, is formally similar to the **complete** tracer equation of the TRM formalism by McDougall and McIntosh (2001) (see their equation (53)). On this basis, some physical oceanographers conclude that solutions of the TRM model coincide with that obtained in ZC-OGCMs used thus far, provided one interprets results for mean tracer fields as that for thickness-weighted mean fields. It would be so if the equation for $\bar{\mathbf{U}}$ were independent from $\bar{\tau}$. In reality the equation depends on density which, in turn, is a function on temperature and salinity. The corresponding term in the momentum equation will change if one changes the interpretation of active $\bar{\tau}$'s. Thus, the form of the momentum equation depends on an interpretation of $\bar{\tau}$ and is different in TRM and Eulerian formalisms that is clear from the TRM momentum equation (66) of MM01 when one rewrites it in terms of $\bar{\mathbf{u}}$. We analyzed the additional terms in the TRM momentum equation and showed that they are essential (Dubovikov and Canuto 2006). In addition, in this work we also showed that the difference of eddy induced velocities in (2a) and TRM is also essential. But anyway, the TRM tracer equation is considerably simpler than the complete equation in ZC formalism. Therefore, TRM might be a good alternative of ZC-OGCMs.

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